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# Particle–antiparticle symmetry of the multi-dimensional fermionic Newton oscillator

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**Abstract**
We consider the $d$-dimensional Newton oscillator, which is invariant under the group $U(d)$, and construct a symmetry operator $R$ which corresponds to particle–antiparticle interchange.

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The development of the quantum inverse method and the study of solutions of the Yang–Baxter equation are the origin of the concept of quantum groups and algebras [1, 2]. It was also found that these new mathematical structures have important applications in exactly solvable statistical models [3] and in two-dimensional conformal field theories [4]. An interesting realization of the quantum algebra $SU_q(2)$ in terms of a $q$-analogue of the usual bosonic harmonic oscillator and the Jordan–Schwinger mapping [5–7] has led naturally to the introduction of a $q$-deformed fermionic equivalent of the $q$-deformed bosonic oscillator to construct the oscillator representation of the $q$-deformed superalgebras [8], quantum exceptional algebras [9] and some $q$-deformed classical Lie algebras [10].

The multi-dimensional fermionic Newton oscillator is defined by the commutation relations

$$\begin{aligned} &a_j a_k^* + q a_k^* a_j = q^N \delta_{jk} \\ &a_j N = (N+1) a_j \\ &a_j a_k + a_k a_j = 0 \\ &j, k = 1, 2, \ldots, d \end{aligned} \tag{1}$$

where $a_j^*$ is the creation operator and $a_j$ is the annihilation operator of the $i$th fermion and $N$ is the total number operator in $d$ dimensions. It enjoys the following properties:

(1) For $q = 1$ it reduces to the usual $d$-dimensional fermion algebra.
(2) The usual fermionic operators $a$ and $a^*$ satisfying

$$\begin{aligned} &aa^* + a^*a = 1 \\ &[N, a^*] = a^* \\ &a^2 = 0 \end{aligned}$$

are equivalent to fermionic $q$-oscillators satisfying

$$aa^* + qa^*a = q^N$$
$$[N, a^*] = a^*$$
$$a^2 = 0.$$

Furthermore, Jing and Xu proved that the well known $q$-deformed fermionic oscillator given by

$$aa^* + a^*a = q^N$$
$$aa^* + q^{-1}a^*a = q^{-N}$$
$$a^2 = a^2 = 0$$

is nothing but the usual fermionic oscillator [11].

Although the fermionic Newton oscillator for $d = 1$ is isomorphic [11, 12] to the usual fermionic oscillator, for $d > 1$ the 'deformed' total particle operator

$$\tilde{N} = \sum_j a_j^* a_j = Nq^{N-1} \tag{2}$$

gives deformed integer eigenvalues.

(3) It is invariant under the $U(d)$ group, which acts on the oscillators through

$$a_j \to \alpha_{jk} a_k \qquad a_j^* \to \bar{\alpha_{jk}} a_k^* \tag{3}$$

where $\alpha_{jk}$ is a $d \times d$ unitary matrix. This property, which is also possessed by the $d$-dimensional classical harmonic oscillator, justifies the name Newton. For the bosonic version it can be shown that the deformation can be understood using the quantization of the classical Newton equation [13].

The coherent states for the one-dimensional bosonic Newton oscillator were constructed in [14]. The multi-dimensional bosonic version of this oscillator was derived in [13] using the quantization of the harmonic oscillator through its Newton equation and its invariance properties. A different multi-dimensional fermionic $q$-oscillator not invariant under $U(d)$ has been considered in [15].

Although the usual $d$-dimensional fermionic oscillator obeys the (particle–antiparticle) symmetry $a_j \leftrightarrow a_j^*$ and $N \leftrightarrow d - N$, the $q$-deformed oscillator does not. In this paper we show that the algebra (1) can be extended such that the system has the symmetry

$$a_j \to b_j \qquad N \to d - N \tag{4}$$

where $b_j$ are 'inverse' lowering operators, and construct a symmetry operator $R$ which interchanges the $a_j$ with the $b_j$. For $q = 1$, $R$ becomes the operator which interchanges $a_j$ and $a_j^*$.

Without loss of generality and to ease the calculations, let us introduce the number operators $N_j$ for $j = 1, 2, \ldots, d$ defined by

$$N_j = q^{1-N} a_j^* a_j. \tag{5}$$

From (1), it follows that $N_j^2 = N_j$ so $N_j$ has eigenvalues 0 and 1 and the total number operator $N$ can be expressed as

$$N = \sum_{j=1}^{d} N_j. \tag{6}$$

With a choice like that in (5) and (6) for the number operator $N_j$, one can easily show the following by using the definitive relations for $N_j$ as well as the commutation relations defined by (1):

$$\begin{aligned}
&N_j a_j = 0 \qquad a_j N_j = a_j \\
&a_j^* N_k = N_k a_j^* \qquad j \neq k \\
&[N_j, N_k] = 0 \qquad N_j^* = N_j \\
&N_j = (N_j)^k \qquad k = 1, 2, \ldots \\
&q^{N_j} = 1 + (q-1)N_j \\
&a_j a_j^* = (1 - N_j) q^{M_j} \qquad a_j^* a_j = N_j q^{M_j} \\
&q^{N_j} a_j^* = q a_j^* \qquad a_j^* = a_j^* q^{N_j}
\end{aligned} \tag{7}$$

where

$$M_j = N_1 + N_2 + \cdots + N_{j-1} + N_{j+1} + \cdots + N_d \tag{8}$$

is an operator with integer eigenvalues.

Before constructing the symmetry operator $R$ interchanging $a_j$ and $a_j^*$, it is worth noting that for $d = 3$ and by setting

$$b_j^* = q^{1-N} a_j \tag{9}$$

the following algebraic relations which both $a_j$ and $b_j$ satisfy are found:

$$\begin{aligned}
&a_j b_k + b_k a_j = q \delta_{jk} \\
&a_j b_k^* = -q^{-1} b_k^* a_j = q^{-1} b_j^* a_k = -a_k b_j^*.
\end{aligned} \tag{10}$$

Similarly the commutation relations which $b_j$ itself satisfies are obtained as follows:

$$\begin{aligned}
&b_j b_k^* + q b_k^* b_j = q^{3-N} \delta_{jk} \\
&b_j^* N = (N+1) b_j^* \\
&b_j b_k = -b_k b_j.
\end{aligned} \tag{11}$$

If the same procedure is carried out for different values of the dimension $d$, it provides that if $b_j^*$ defined by (9) for $d = 3$ is replaced by

$$b_j^* = q^{(d-1-2N)/2} a_j \tag{12}$$

the structure of the commutation relations can be generalized for any $d$ by

$$\begin{aligned}
&b_j b_k^* + q b_k^* b_j = q^{d-N} \delta_{jk} \\
&b_j^* N = (N+1) b_j^* \\
&b_j b_k = -b_k b_j
\end{aligned} \tag{13}$$

with

$$\sum_{j=1}^{d} b_j^* b_j = (d - N) q^{d-1-N}. \tag{14}$$

By a similar procedure, it can easily be shown that the commutation relations between $a_j$ and $b_j$ are given by

$$\begin{aligned}
&b_j^* a_k^* + a_k^* b_j^* = q^{(d-1)/2} \delta_{jk} \\
&a_j b_k^* = -q^{-1} b_k^* a_j = q^{-1} b_j^* a_k = -a_k b_j^*.
\end{aligned} \tag{15}$$

This algebraic structure can then be used to solve the problem of constructing a symmetry operator, denoted by $R$, which interchanges the $a_j$ with the $b_j$. Such an operator $R$ can be characterized by

$$\begin{aligned} &R = R^* = R^{-1} \\ &Ra_jR = \varphi_j b_j \\ &RNR = d - N \end{aligned} \qquad (16)$$

where the operator $\varphi_j$ is a phase which commutes with $b_j$ and turns out to be given by

$$\begin{aligned} &\text{for} \quad d = 2k+1 \qquad \varphi_j = e^{i\pi M_j} \\ &\text{for} \quad d = 4k+4 \qquad \varphi_j = \left\{ \begin{array}{ll} e^{i\pi M_j} & M_j < \frac{d}{2} \\ -e^{i\pi M_j} & M_j \geqslant \frac{d}{2} \end{array} \right\} \\ &\text{for} \quad d = 4k+2 \qquad \varphi_j = \left\{ \begin{array}{cc} i & \frac{d}{2} - 1 \leqslant M_j \leqslant \frac{d}{2} \\ e^{i\pi M_j} & M_j < \frac{d}{2} - 1 \\ -e^{i\pi M_j} & M_j > \frac{d}{2} \end{array} \right\} \end{aligned} \qquad (17)$$

with $M_j$ as defined in (8). It should be noted that $R$ will depend on the dimension $d$ but not on the index $j$ of the oscillator creation and annihilation operators $a_j$ and $a_j^*$.

The definition above of the symmetry operator $R$ will lead to the following relations:

$$\begin{aligned} &RN^kR = (d - N)^k \\ &R(N + k)R = (d + k) - N. \end{aligned} \qquad (18)$$

The equations (16) and (18) for the symmetry operator $R$ as well as the creation, annihilation and 'inverse' lowering operators suggest that for a two-dimensional system $R = R(2)$ can be characterized as

$$R(2) \equiv \alpha(a_1^* a_2^* + a_2 a_1) + \beta(a_j a_1^* a_2^* a_j - a_j^* a_2 a_1 a_j^*) \qquad (19)$$

where the Einstein summation convention over repeated indices holds and the coefficients $\alpha$ and $\beta$ are to be determined.

If the ground state is denoted by $|\ \rangle$ and the state $|1, 2\rangle$ by $|\bar{\ }\rangle$, by defining

$$\gamma|\bar{\ }\rangle = a_1^* a_2^* |\ \rangle = a_1^*|2\rangle$$

one finds that $\gamma = q^{1/2}$ by simply calculating $\gamma$ from the following equation:

$$|\gamma|^2|\bar{\ }\rangle = \langle\ |a_2 a_1 a_1^* a_2^*|\ \rangle.$$

In general,

$$|\gamma|^2\langle 1, 2, \ldots, d|1, 2, \ldots, d\rangle = \langle\ |a_d a_{d-1} \cdots a_2 a_1 a_1^* a_2^* \cdots a_{d-1}^* a_d^*|\ \rangle$$

which leads to

$$\gamma = q^{d(d-1)/4} \qquad (20)$$

for all dimensions; that is

$$q^{d(d-1)/4}|1, 2, \ldots, d\rangle = a_1^* a_2^* \cdots a_d^*|\ \rangle.$$

By imposing

$$\begin{aligned} &R|\ \rangle \sim |\bar{\ }\rangle \\ &R|\bar{\ }\rangle \sim |\ \rangle \end{aligned}$$

and

$$\begin{aligned} &R|1\rangle \sim |2\rangle \\ &R|2\rangle \sim |1\rangle \end{aligned}$$

it is found that $\alpha$ must be equal to $q^{-1/2}$ by using the first set whereas the action of $R$ on $|1\rangle$ and $|2\rangle$ should be modified by phases such that

$$R|1\rangle = -\mathrm{i}|2\rangle$$
$$R|2\rangle = +\mathrm{i}|1\rangle$$

to determine $\beta$. With this modification $R$ is calculated to be

$$R(2) \equiv q^{-1/2}(a_1^* a_2^* + a_2 a_1) + q^{-1}\frac{\mathrm{i}}{2}(a_j a_1^* a_2^* a_j - a_j^* a_2 a_1 a_j^*).$$

It is important to note that complex coefficients only appear when $d = 4k + 2$, where $k = 0, 1, 2, \ldots$.

For simplification in the formulation of $R$, let us define operators $A_d$ and $B_k$ as follows:

$$\begin{aligned}
A_d^*(a) &= a_1^* a_2^* \ldots a_d^* = \prod_{j=1}^{d} a_j^* \\
A_d(a) &= [A_d^*(a)]^* = \prod_{j=1}^{d} a_{d-j+1} \\
A_d^*(a_l) &= a_{l_1}^* a_{l_2}^* \ldots a_{l_d}^* = \prod_{j=1}^{d} a_{l_j}^* \\
A_d(a_l) &= [A_d^*(a_l)]^* = \prod_{j=1}^{d} a_{l_{d-j+1}} \\
A_0(a) &= 1 \\
B_k &= A_k(a_l) A_d^*(a) A_k(a_l) \\
B_k^* &= A_k^*(a_l) A_d(a) A_k^*(a_l).
\end{aligned} \tag{21}$$

By repeating similar calculations for different dimensions, the formula for the symmetry operator can be generalized to obtain

$$R(d) = S_d + \sum_{k=0}^{[d/2-1]} \frac{1}{k!} q^{-(d-1)(d/2+k)/2} \, (B_k + B_k^*) \tag{22}$$

where

$$S_d = \left\{ \begin{array}{ll} 0 & d = 2k+1 \\ \frac{\mathrm{i}}{2}\frac{1}{(d/2)!} q^{-d(d-1)/2}(B_{d/2} - B_{d/2}^*) & d = 4k+2 \\ \frac{1}{2}\frac{1}{(d/2)!} q^{-d(d-1)/2}(B_{d/2} + B_{d/2}^*) & d = 4k+4 \end{array} \right\}. \tag{23}$$

The verification of the properties of $R$ as given by (16) is rather tedious but relatively straightforward. We would like to recall that we have defined the operators $b_j$, $b_j^*$ by (12) and have constructed $R$ so that (16) is valid. In particular this definition implies the existence of the phases $\varphi_j$ in (16). Alternatively (16) could be replaced by a similar set of relations with $\varphi_j = 1$ but then nontrivial phases would be present in (12).

We have shown that the algebra given by (1), (13) and (15) has a symmetry operator $R$ which can be used to implement the symmetry associated with the reflection $a_j \leftrightarrow b_j$. When a classical system is quantized, the preferred method is to choose a scheme where the symmetries of the system remain unchanged. The $d$-dimensional bosonic Newton oscillator and the fermionic Newton oscillator both enjoy the $U(d)$ symmetry of the classical oscillator. Thus together with the particle–antiparticle interchange symmetry we have shown here, the resemblance of the $d$-dimensional fermionic Newton oscillator to the standard $d$-dimensional fermionic oscillator becomes manifest. This symmetry will be relevant for physical models utilizing the fermionic Newton oscillator.

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